Fixed point analysis of a scalar theory with an external field

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A momentum dependent projection of the Wegner-Hougton equation is derived for a scalar theory coupled to an external field. This formalism is useful to discuss the phase diagram of the theory. In particular we study some properties of the Gaussian fixed point.

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In field theory there are several physical situations where the scalar self-interacting theory plays a role of paramount importance. In some cases it is useful to introduce the order parameters, lorentz-scalar from which one can extract informations on the ground state properties when macroscopic quantities like the temperature or the density are changed. In QCD the representative examples are given by the two flavour singlet quantities namely the Polyakov loop and the trace of the fermion condensate. The first signals the appearance of the deconfining transition and the second one the chiral symmetry restoration.

On the other hand, the scalar theory has a central role in the Standar Model and GUT's. Therefore it is of great interest to have a careful understanding of its fixed point structure in order to determine the phase diagram and all possible relevant directions in the parameter space.

In the framework of the ϵ-expansion for the scalar theory, one finds only the Gaussian fixed point in any dimensions and the Wilson-Fisher fixed point for $2 < D < 4$ [1]. An interesting issue has been recently raised in [2] where it is found that around the Gaussian fixed point new relevant directions exist; this result however is not widely accepted [3].

A convenient way to discuss these problems is provided by the Wegner-Houghton equation [4] which is obtained by performing a functional integration over the infinitesimal momentum shell $k, k - \delta k$ in the limit $\delta k \to 0$. The above functional equation, which in principle is exact and contains the whole effect due to the integration of the fast modes, has no known exact analytical solution and one has to resort to approximations in order to deal with a more tractable problem. In [5] a momentum independent projection of the Wegner-Houghton equation has been used to obtain a renormalization group equation for the local potential, and it has been used to classify the interactions around the Gaussian fixed point. Later the $O(\partial^2)$ contributions have been included and the anomalous dimension has been calculated [6].

In this Report we use the gradient expansion in order to project on a larger subset of the parameter space by including a generic field dependence in the wavefunction renormalization function [7,8]. We consider a system of two coupled scalar fields ϕ and ψ, where the second one is treated as classical, i.e. its quantum fluctuations are neglected. This framework is useful when one is interested in studying the phase diagram for the field ϕ in the presence of an external field ψ, whose functional form is kept fixed, which in practice is realized if the cut-off of the integrated field is much below the mass of the other (external) field. In fact another application is a problem where a mass hyerarchy is present. For example in some cases one is interested in integrating out a heavy field in order to construct the effective theory in the low energy domain. In these cases it is important to monitor the decoupling of the heavy field from the low energy effective theory. Conversely one can discuss the effects on a heavy field of the integration of an extremely light one. Here we concentrate on the first issue and analyse how some properties of the fixed point structure of a scalar field are modified by turning on a new scalar degree of freedom, which we treat as classical.

We start with the euclidean action defined at the cut-off k

$$S_k[\phi,\psi] = \int d^Dx \Big(\frac{1}{2} Z_k(\phi,\psi)\partial_\mu\phi\partial^\mu\phi + U_k(\phi,\psi) + \frac{1}{2} W_k(\phi,\psi)\partial_\mu\psi\partial^\mu\psi + Y_k(\phi,\psi)\partial_\mu\phi\partial^\mu\psi\Big). \tag{1}$$

The renormalization group equation is obtained by means of the blocking transformation applied to the ϕ field. By performing the functional integration in the path integral only on the components of the ϕ field with momentum within the shell $(k, k - \delta k)$, we obtain the blocked action

$$\begin{aligned} S_{k-\delta k}[\Phi,\Psi] = \int d^Dx \Big(&\frac{1}{2} Z_{k-\delta k}(\Phi,\Psi)\partial_\mu\Phi\partial^\mu\Phi + \\ &\frac{1}{2} W_{k-\delta k}(\Phi,\Psi)\partial_\mu\Psi\partial^\mu\Psi + Y_{k-\delta k}(\Phi,\Psi)\partial_\mu\Phi\partial^\mu\Psi + \\ &U_{k-\delta k}(\Phi,\Psi)\Big) = \\ &S_k[\Phi,\Psi] + \frac{1}{2}\mathrm{Tr}'\ln K - \frac{1}{2}(FK^{-1}F) \end{aligned} \tag{2}$$

where

$$F = (\delta S_k/\delta\phi)|_{\phi=\Phi,\psi=\Psi} \quad K = (\delta^2 S_k/\delta\phi\delta\phi)|_{\phi=\Phi,\psi=\Psi} \tag{3}$$

and the prime means that the trace is taken over the infinitesimal shell $(k, k-\delta k)$. We use the gradient expansion in order to calculate the trace in the above expression. We set $\Phi(x) = \phi_0 + \tilde{\phi}(x)$ and $\Psi(x) = \psi_0 + \tilde{\psi}(x)$ and we write $K = K_0 + \delta K$ where $K_0 = K(\phi_0, \psi_0) = Z_k(\phi_0, \psi_0)p^2 + \partial^2_\phi U_k(\phi_0, \psi_0)$, the momentum p lies in the shell, and δK is quadratic in both $\tilde{\phi}$ and $\tilde{\psi}$. Then, the following expansion is used

$$\mathrm{Tr}'\ln K = \mathrm{Tr}'\Big(\ln K_0 + K_0^{-1}\delta K - \frac{1}{2}K_0^{-1}\delta K K_0^{-1}\delta K + O(\delta K^3)\Big) \quad (4)$$

and we compute the trace in the last two terms by retaining at the same time operators in the momentum and coordinate representation, as long as all the operators in one representation are on the right side of the operators in the other representation. In the last term of the expression this order is not fulfilled and we use commutation rules to obtain the right ordering. All required commutators can be deduced from the simple rule $[f(x), p_\mu] = -i\partial_\mu f(x)$. Once the $p-x$ dependence has been disentangled, one can perform the trace operation and identify in the right hand side of Eq.(4) contributions to the local blocked potential $U_{k-\delta k}$ and to the wavefunction renormalization functions $Z_{k-\delta k}, W_{k-\delta k}, Y_{k-\delta k}$. Had we started with the action

$$S_k[\phi,\psi] = \int d^Dx\Big(-\frac{1}{2}z_k(\phi,\psi)\phi\Box\phi - \frac{1}{2}w_k(\phi,\psi)\psi\Box\psi + y(\phi,\psi)\partial_\mu\phi\partial^\mu\psi + U(\phi,\psi)\Big) \quad (5)$$

the functions z, w, y could have been expressed in terms of Z, W, Y, previously introduced, through $Z = \partial_\phi(\phi z)$, $W = \partial_\psi(\psi w)$ and $Y = y + [\phi\partial_\psi z + \psi\partial_\phi w]/2$, so there is no ambiguity in the method.

After a rather long but straightforward calculation one finds the following flow equations for the dimensionless renormalized blocked potential $V = Uk^{-D}$ (the subscript k will be omitted from now on and we define $t = ln(k/\Lambda) < 0$ where Λ is a fixed UV scale)

$$\frac{\partial V}{\partial t} = \frac{(D-2)}{2}(x\partial_x V + y\partial_y V) - DV - a_D\ln\Big(\frac{\mathcal{A}}{\mathcal{A}|_{x=y=0}}\Big) \quad (6)$$

and the wavefunction renormalization functions

$$\frac{\partial Z}{\partial t} = \frac{(D-2)}{2}(x\partial_x Z + y\partial_y Z) - a_D\mathcal{A}_t^{-1}\Big(\partial_x^2 Z - 2\mathcal{A}^{-1}\partial_x Z\partial_x\mathcal{A} + \mathcal{A}^{-2}Z\partial_x\mathcal{A}[\partial_x\mathcal{A} + \partial_x Z] - \mathcal{A}^{-3}Z^2(\partial_x\mathcal{A})^2\Big) \quad (7)$$

$$\frac{\partial Y}{\partial t} = \frac{(D-2)}{2}(x\partial_x Y + y\partial_y Y) - a_D\mathcal{A}^{-1}\Big(\partial_x^2 Y - \mathcal{A}^{-1}(\partial_x Z\partial_y\mathcal{A} + \partial_x Y\partial_x\mathcal{A}) + \mathcal{A}^{-2}Z(\partial_x\mathcal{A}\partial_y\mathcal{A} + \partial_y\mathcal{A}\partial_x Z/8 + \partial_x\mathcal{A}\partial_y Z/8) - \mathcal{A}^{-3}Z^2\partial_x\mathcal{A}\partial_y\mathcal{A}/8\Big) \quad (8)$$

$$\frac{\partial W}{\partial t} = \frac{(D-2)}{2}(x\partial_x W + y\partial_y W) - a_D\mathcal{A}\Big(\partial_x^2 W - 2\mathcal{A}^{-1}\partial_x Y\partial_y\mathcal{A} + Z\mathcal{A}^{-2}\partial_y\mathcal{A}(\partial_y\mathcal{A} + \partial_y Z) - Z^2\mathcal{A}^{-3}(\partial_y\mathcal{A})^2\Big) \quad (9)$$

where $\mathcal{A}(t,x,y) = Z(t,x,y) + \partial_x^2 V(t,x,y)$ $a_D = 1/2^D\pi^{D/2}\Gamma(D/2)$ and we have introduced the dimensionless variables $x = \Phi k^{-(D-2)/2}, y = \Psi k^{-(D-2)/2}$.

The above set of equations has been used to discuss the Decoupling Theorem (DT) [9] within this approach in [10]. In particular, making use of a polynomial expansion of V and Z in the two fields, it has been shown that no deviation from the DT appears in this context but around the heavy mass threshold some small effect of non-locality is present in the wave function renormalization Z.

Here, since we do not expect large effects due to W and Y, we just consider, for the analysis of the fixed point structure, the two coupled equations (6),(7) in the case $2 < D \le 4$. We begin our investigation by perturbing a fixed point of the theory with $y = 0$, with functions that in general depend on y.

Let us first consider the Gaussian fixed point, solution of Eqs. (6),(7),

$$V^* = 0, \quad Z^* = \text{const.} = \mathrm{Z}_0 \quad (10)$$

We thus write $V = \delta V$ and $Z = Z_0 + \delta Z$ where the perturbed quantities depend on the external field and linearize the flow equations. From (6),(7) we obtain

$$\frac{\partial\delta V}{\partial t} = \frac{(D-2)}{2}(x\partial_x\delta V + y\partial_y\delta V) - D\delta V - \alpha(\delta Z + \partial_x^2\delta V) + C \quad (11)$$

$$\frac{\partial\delta Z}{\partial t} = \frac{(D-2)}{2}(x\partial_x\delta Z + y\partial_y\delta Z) - \alpha\partial_x^2\delta Z \quad (12)$$

where $C = \alpha(\delta Z + \partial_x^2\delta V)|_{x=y=0}$ and $\alpha = a_D/Z_0$. In order to study the quantized-λ behavior, we assume for the perturbations the general form

$$\delta V = e^{-\lambda t}h(x)g(y) \qquad \delta Z = e^{-\lambda t}s(x)r(y), \quad (13)$$

therefore, since t is non-positive, $\lambda > 0$, $= 0$, < 0 correspond respectively to relevant, marginal or irrelevant directions. Thus one obtains the eigenvalue equations

$$\frac{D-2}{2}(g(y)x\partial_x h(x) + h(x)y\partial_y g(y)) - Dh(x)g(y) - \alpha(g(y)\partial_x^2 h(x) + s(x)r(y)) + C_0 = -\lambda h(x)g(y) \quad (14)$$

$$\frac{D-2}{2}(r(y)x\partial_x s(x) + s(x)y\partial_y r(y)) - \alpha r(y)\partial_x^2 s(x) = -\lambda s(x)r(y) \quad (15)$$

where $C_0 = e^{\lambda t}C$. The solution of these two coupled equations, for any positive integer m, non-negative integer n and arbitrary integration constant $C_2 \neq 0$, is

$$\delta V(x,y) = -\frac{\alpha}{D} C_2 e^{-\lambda_{mn} t} H_n(\beta x) y^m \tag{16}$$

$$\delta Z(x,y) = C_2 e^{-\lambda_{mn} t} H_n(\beta x) y^m \tag{17}$$

where

$$\lambda_{mn} = -\frac{m+n}{2}(D-2) \tag{18}$$

and $H_n(\beta x)$ are the Hermite polynomials and $\beta = \sqrt{(D-2)/(4\alpha)}$.

Eq. (18) shows that for the values chosen for m and n, all eigen-directions around the fixed point are irrelevant.

It must be remarked that, since we are in the presence of an external field, we retain solutions with odd m and n, corresponding to unbounded potentials; obviously, when a specific problem is considered, one must reject all solutions that are unphysical for that particular case.

Negative integer values of m instead give bounded solutions that, however, are singular in $y = 0$. We shall comment on these later.

A slightly different solution, involving an additive constant in the potential δV, is obtained by selecting $m = 0$ and, again, any non-negative integer n in the eigenvalue (18)

$$\delta V = -\frac{\alpha}{D} C_2 e^{-\lambda_{0n} t}(H_n(\beta x) - H_n(0)) \tag{19}$$

$$\delta Z = C_2 e^{-\lambda_{0n} t} H_n(\beta x) \tag{20}$$

Obviously the latter case corresponds to a decoupling of the field y which does not appear in the solutions, and again all perturbations are irrelevant except if $n = 0$ where the interaction is instead marginal. However, since $H_0(\beta x)$ is a constant, it is easy to realize the $n = 0$ solution is nothing else than the Gaussian fixed point (10), which explains the marginality of the solution.

The set of solutions displayed so far is not complete since the two equations (11), (12) decouple for the particular choice $\delta Z = 0$. In this case the equation for δV only has to be solved and this yields new solutions which read (n is again any non-negative integer and C_2 an arbitrary non-vanishing integration constant)

$$\delta V = e^{-\lambda_{mn} t} C_2 y^m H_n(\beta x), \tag{21}$$

with positive integer m, and

$$\delta V = e^{-\lambda_{0n} t} C_2 (H_n(\beta x) - H_n(0)) \tag{22}$$

with $m = 0$. This time λ_{mn} is

$$\lambda_{mn} = -\frac{(m+n)}{2}(D-2) + D \tag{23}$$

which, for the latter case with $m = 0$ gives the well known classification of the polynomial interactions when y is turned off.

Summarizing, any perturbation in the external field which is present in the wavefuntion renormalization function δZ, also appears in the potential and provides only irrelevant directions in the parameter space with eigenfunctions that behave like $H_n y^m$ where H_n is a rescaled Hermite polynomial. The case corresponding to a constant non-vanishing δZ is marginal, but the solution is just the Gaussian fixed point. Finally, for $\delta Z = 0$, the equation for δV decouples and we have new marginal and relevant perturbations due the presence of the external field. In particular, we note that in $D = 4$ the coupling corresponding to the mass of the external field is a relevant interaction, although no integration on the external field modes has been performed in the blocking procedure.

In this analysis we have not included the solutions with negative integer values of m, that can correspond to positive λ_{mn} in Eq. (18) and thus to relevant directions, but are singular in $y = 0$. They can be formally taken into account, provided one does not turn off the field y and replaces the normalization introduced in Eq. (11) through the constant C, which would be ill defined, with another one that is suitable for this case. However the limit where the influence of the external field becomes small and then disappears is now reached for $y \to \infty$, whereas for $y << 1$ the physics of the field x is strongly modified by the other field y. One can think of couplings like x/y in the lagrangian that could play such a role, but we do not see any clear physical meaning that can be attached to these solutions.

The search for non-trivial fixed points is a difficult task due to the non-linearity of the RG equations. For $2 < D < 4$ one finds the ferromagnetic fixed point with only one relevant direction. The asymtpotic form of the potential for large field in this fixed point can be found in [6] and, in principle, it is possible to study the modifications of the eigen-directions in the presence of an external field, as we have done for the gaussian fixed point, through a numerical investigation of equations (6), (7).

From a different point of view our equations can be used to study the t invariant solutions of (6), (7) with the external field turned on. In this case the external field acts on the system as a new scale. In fact Eq. (6) with the left hand side equal to zero, can be recast in the form

$$\frac{\partial V^*}{\partial \Lambda} = \frac{(D-2)}{2} x \partial_x V^* - D V^* - a_D \ln\left(\frac{\mathcal{A}^*}{\mathcal{A}^*|_{x=y=0}}\right) \tag{24}$$

with $-\Lambda = (2/(D-2))\ln y$ where it is manifest the role played by the external field. For instance, if a solution of Eq. (24) exists, it is determined by the strength of the external field and by three arbitrary constants α, β and γ. Its asymptotic form, for large x is

$$Z^* = \alpha + \frac{y^2}{x^2} + +O(x^{-8}) \tag{25}$$

$$V^* = \beta x^6 + y^2 x^4 + \gamma y^4 x^2 - \frac{a_D}{3} \ln \frac{30\alpha\beta}{\mathcal{A}^*|_{x=y=0}} - \frac{4a_D}{3}\ln x - 2a_D/9 - \frac{a_D}{150\beta x^4} - \frac{2y^2}{15\beta x^2} + O(x^{-6}) \tag{26}$$

and we have supposed $y \ll x$ and we have worked out the expressions up to $O(y^4/x^4)$ terms. An interesting question is whether or not one recovers the Wilson fixed point in $y = 0$ starting from a non-zero external field strength as in Eqs. (25),(26). If this is not the case one should question the uniticity of the solution of the fixed point equations in $y = 0$. A numerical investigation of the equations is mandatory in order to answer this issue. We hope to address this point in a following communication.

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